

Comparison of Current Gravity Estimation and Determination Models

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Abstract

This paper will discuss the history of gravity estimation and determination models while analyzing methods that are in development. Some fundamental methods for calculating the gravity field include spherical harmonics solutions, local weighted interpolation, and global point mascon modeling (PMC). Recently, high accuracy measurements have become more accessible, and the requirements for high order geopotential modeling have become more stringent. Interest in irregular bodies, accurate models of the hydrological system, and on-board processing has demanded a comprehensive model that can quickly and accurately compute the geopotential with low memory costs. This trade study of current geopotential modeling techniques will reveal that each modeling technique has a unique use case. It is notable that the spherical harmonics model is relatively accurate but poses a cumbersome inversion problem. PMC and interpolation models, on the other hand, are computationally efficient, but require more research to become robust models with high levels of accuracy. Considerations of the trade study will suggest further research for the point mascon model. The PMC model should be improved through mascon refinement, direct solutions that stem from geodetic measurements, and further validation of the gravity gradient. Finally, the potential for each model to be implemented with parallel computation will be shown to lead to large improvements in computing time while reducing the memory cost for each technique.

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1. Introduction

1.1 Motivation

Accurate calculation of the geopotential has become more desirable as requirements for precise orbit determination, geodesy, hydrology, and interest in the exploration of irregular bodies has grown. Recently, high quality data from missions like GRACE and GOCE have enabled ultra-high-fidelity models of the global gravity field, but they have also spurred development in our techniques for modeling the geopotential [1].

Typically, our techniques for modeling the geopotential have been limited by the memory space available on processors. However, the speed and size of computer chips has increased dramatically, making higher order gravity fields computationally feasible. Remarkably, a gravity field of full degree/order 360 can be calculate with a single CPU. As memory has become more accessible, it has become more feasible to trade memory for speed for many applications. Due to the limited portability of ultra-high-performance computing machines, this paper will generally focus on optimizing software implementations for common hardware applications.

However, this study will suggest a hardware implementation where many orders of magnitude of computational speedup can be achieved through the efficient use of parallel computation methods using inexpensive, off the shelf, GPUs. The gravity estimation problem is well suited for parallelism, and the hardware necessary for parallel implementation is far from restrictive.

The most computationally expensive calculations required for high fidelity simulations of spacecraft motion lie within the evaluation of local gravity components, geomagnetic fields, and space weather prediction. This paper will focus on methods that seek to alleviate the burden of

computing the gravity field. The largest computational expenses in the gravity field estimation are defined by Daniel Oltrogge [2] as follows:

1. Evaluating and performing coordinate transformations
2. Evaluating planetary and other third body positions
3. Evaluating accelerations acting upon the satellite

The models presented in this paper will aim to simplify the burden of evaluating attitudinal and positional accelerations that act on the satellite. More specifically, efficient methods for estimating and evaluating the gravity field will be explored.

This research is motivated by movements in the field of astrodynamics that require more efficient methods for estimating and evaluating the gravity field. High Fidelity gravity estimation and evaluation will be required for:

- precise orbit determination (P.O.D)
- Accurate mapping of Earth's hydrological variations
- Monte Carlo simulations for trajectory planning

Furthermore, high speed calculations are increasingly important for applications including:

- Real time P.O.D
- Computationally intensive Monte Carlo simulations
- Increasing numbers of objects tracked by space catalogs

Additionally, asteroid missions have become a top priority for the aerospace community over the last few decades. These missions present unique challenges for orbit estimation and determination due to the irregular shape of these bodies and the lack of a-priori knowledge

concerning their gravity field. The spherical harmonics approach is not well suited for this application for several reasons: it provides global representations without direct local representations, it is much too slow to compute real-time (as required without a-priori knowledge), and series expansions of orthogonal expansions are not well suited for objects that are not approximately spherical or ellipsoidal.

1.2 Thesis Outline

This report aims to summarize the current state of geopotential modeling. This paper will recognize the four categories of gravity representations outlined by Tschering (1996) [3]:

1. Series Expansion of Orthogonal Functions
2. Linear Combinations of Potential Functions
3. Linear Combinations of Functions defined via splines, kernels, or finite elements
4. Collocation methods using minimum norm or least squares

Each modeling method will be introduced with a brief overview of theory, followed by an analysis of various implementations within the category. When available, recent implementations within each category will be used to assess the relative strengths and weaknesses of the evaluation technique. This paper aims to assess the overall performance of current gravity estimation and evaluation techniques by comparing the relative speed, accuracy, and complexity of each model. In doing so, it is recognized that some models are well suited for applications that others are not. In concluding this paper, the best use case for each technique will be defined and suggestions will be offered for the next generation of research.

2. Background

2.1 Geoid and Geopotential

The geoid is an equipotential surface that accounts for the non-uniform distribution of Earth's mass. The model corresponds to the theoretical shape the oceans take under the influence of gravity and rotation, alone. This surface can be defined at any potential, resulting in geopotential values defined at all heights. To calculate the geoid, extensive measurements and calculations are necessary. The geoid is a useful feature in Earth system sciences because the gravitational acceleration is perpendicular to the geoid at all points along its surface.

2.2 Spherical Harmonics Standard

The classical method for estimating the geopotential uses a linear combination of an infinite series of orthogonal functions along a sphere. Using a set of solutions to the Laplace equation, the Stokes coefficients for the spherical harmonics series are used to estimate the gravity field.

The current demands of astrodynamics require a more efficient computational method for gravity estimation. Spherical harmonics are computationally burdensome at high degree/order, and they occupy a large memory footprint. In fact, increasing the desired accuracy of the SH solution produces a quadratic increase in the computational requirements for the solution. Furthermore, spherical harmonics required ill-conditioned inversions and often use recursive relationships that are not amenable to parallelism.

Theory

3. Series Expansions of Orthogonal Functions

The basis for computing series expansions lies in a fundamental relationship defined by Laplace. Laplace's famous equation is as follows, where U is the potential:

$$\nabla^2 U = 0,$$

The solution to Laplace's equation above can be computed using Thomson and Tait's Legendre polynomials developed in 1879 [4]. The polynomials use boundary values on the surface of the sphere to produce a solution that includes the nonzero order terms which define variations in gravity with longitude. In practice, increasing the degree of the spherical harmonic decreases the wavelength of the solution set. In doing so, the difficulty of evaluating the Legendre functions increases exponentially. Most modern solutions for spherical harmonics utilize recursive formulations, and in some cases, Fourier transformations are employed to compute solutions in the frequency domain.

3.1 Spherical Harmonics

Traditionally, the geopotential is estimated using a series of spherical harmonics. The spherical harmonics equation is solved as a set of solutions to the Laplace equation. Each of Stokes coefficients is determined through an iterative method that represents the gravity field. The static geopotential field can be represented in the following series:

$$W_a(r, \lambda, \varphi) = \frac{GM}{r} \sum_{\ell=0}^{\ell_{\max}} \sum_{m=0}^{\ell} \left(\frac{R}{r} \right)^{\ell} P_{\ell m}(\sin \varphi) (C_{\ell m}^W \cos m\lambda + S_{\ell m}^W \sin m\lambda) \quad (108)$$

which shows the $1/r$ -behaviour for $r \rightarrow \infty$, or written in the form

$$W_a(r, \lambda, \varphi) = \frac{GM}{R} \sum_{\ell=0}^{\ell_{\max}} \sum_{m=0}^{\ell} \left(\frac{R}{r} \right)^{\ell+1} P_{\ell m}(\sin \varphi) (C_{\ell m}^W \cos m\lambda + S_{\ell m}^W \sin m\lambda)$$

which is sometimes useful in practice. The notations are:

- r, λ, φ - spherical geocentric coordinates of computation point
(radius, longitude, latitude)
- R - reference radius
- GM - product of gravitational constant and mass of the Earth
- ℓ, m - degree, order of spherical harmonic
- $P_{\ell m}$ - fully normalised Legendre functions
- $C_{\ell m}^W, S_{\ell m}^W$ - Stokes' coefficients (fully normalised)

Figure: Spherical Harmonics Equations [5]

The degree and order of the system can be chosen to improve the resolution of the spherical harmonics solution. Typically, spherical harmonics are solved for a degree/order of either 180 or 360. As shown in the table below, the number of coefficients required to solve the spherical harmonic grows exponentially with degree and order. However, current harmonic solutions at degree/order of 180 and 360 have proven to be remarkably accurate representations of the geopotential.

Maximum Degree	Number of Coefficients	Resolution ψ_{min}			
		eq. (112)		eq. (114)	
ℓ_{max}	N	[degree]	[km]	[degree]	[km]
2	9	90.000	10000.000	77.885	8653.876
5	36	36.000	4000.000	38.376	4264.030
10	121	18.000	2000.000	20.864	2318.182
15	256	12.000	1333.333	14.333	1592.587
30	961	6.000	666.667	7.394	821.587
36	1369	5.000	555.556	6.195	688.321
40	1681	4.500	500.000	5.590	621.154
45	2116	4.000	444.444	4.983	553.626
50	2601	3.600	400.000	4.494	499.342
75	5776	2.400	266.667	3.016	335.073
180	32761	1.000	111.111	1.266	140.690
360	130321	0.500	55.556	0.635	70.540
500	251001	0.360	40.000	0.457	50.828
1000	1002001	0.180	20.000	0.229	25.439
2000	4004001	0.090	10.000	0.115	12.726
5000	25010001	0.036	4.000	0.046	5.092
10000	100020001	0.018	2.000	0.023	2.546

Figure: Spatial Resolution of spherical harmonics in terms of smallest representable shape [5]

Satellites like GOCE and GRACE, which use laser altimetry to provide remarkably precise measurements of the geopotential have provided the means for much more accurate gravity representations. In fact, the GGM03S GRACE model [1] uses four years of data to model the geopotential up to degree 180. When paired with surface gravity and altimetric mean sea surface measurements, the GGM03C model is useful to degree 360, while including a covariance matrix.

The GGM03S model uses four years of data to model the geopotential up to degree 180. When this data is combined with surface gravity and altimetric mean sea surface measurements, it can provide a model (GGM03C) that is useful to degree 360 and includes a covariance matrix. The model obtains remarkable accuracy through a processing strategy that allows for the weighting of inhomogeneous data types. However, this model demonstrates the difficulty of spherical harmonic computation: GGM03C requires 130,000 parameters that occupy 68 GB of space.

The major advantage of the spherical harmonics approach is that it is guaranteed to converge at all points that lie outside of the circumscribing sphere. The series can be truncated at any finite order to achieve desired resolution and accuracy of the global field. However, this truncation is simply an approximation rather than a perfect representation of the field.

In their calculation of the RL05 Mascon field, Russell and Arora note the weaknesses of spherical harmonics as a means for mapping the geoid [7]. Their research argued that high degree harmonics have large errors in the estimation of the Stoke's coefficients due to poor

observability in the East to West direction. Further, they argue that the ill-posed inversion problem makes spherical harmonics solutions inherently slow. Finally, they noted that unconstrained GRACE solutions have suffered from north-south stripes. Although these stripes are traditionally removed using de-striping filters, they often produce physically unrealistic regions of the geoid model.

Although spherical harmonics have proven to be effective as accurate solutions for the geopotential, they are computationally burdensome, and they are recursive. High-fidelity computation of the geopotential using spherical harmonics is time intensive which makes it difficult to compute in real-time applications.

Spherical harmonics suffer many drawbacks when studying irregular bodies. They often diverge inside of their circumscribing sphere, and they present no information about whether a point is inside or outside of a body. For orbital trajectory computation, a separate computation is required to ensure that orbits do not lead to collisions with the body. Finally, spherical harmonics present large errors close to the radius of convergence. In fact, spherical harmonics were shown to exhibit errors larger than 100% when studying 4769 Castalia [8].

Sucarrat and Palmer [9] offer a straightforward solution for spherical harmonics inside of the circumscribing sphere. Whereas spherical harmonics and spherical Bessel functions satisfy Laplace's equation outside of the circumscribing sphere, they argue that Poisson's equation can be effectively applied to provide SH solutions inside of the Brillouin sphere.

In a simple model of 4769 Castalia [8], using an elongated body up to second order, SH solutions for Laplace converged outside of the shell and Laplace's equation was satisfied near the surface with no divergence.

4. Linear Combinations of Potential Functions

4.1 Polyhedron Modeling

Recent interest in the study of comets and asteroids has prompted a necessity for calculating and modeling the geopotential of highly irregular bodies. The advantage of polyhedron modeling of surfaces is that they will converge everywhere (unlike the spherical harmonics model). The polyhedron model attempts to remedy the drawbacks of the spherical harmonic and mascon models. Polyhedron modeling allows for convergence everywhere – inside and outside of the surface. It also provides an exact solution for the gravity field (excluding the errors from discretization). Finally, an evaluation of the Laplacian of gravitational potential allows for the determination of whether a point is inside or outside of the body.

The CSR RL05 mascon model [10] defines a geodesic grid that models Earth's surfaces as set of tiles created formed by a subdivided icosahedron inside of a circumscribing sphere (brillouin). This icosahedron iteratively bisected into four equal triangles until a polyhedron with 40962 vertices is formed. The resultant surface has 40950 hexagonal tiles and 12 pentagonal tiles. This choice provides a resolution of 1 degree at the equator such that each tile has an area of 12400 km.

Similarly, a volume model uses a set of cubes or spheres to model irregular bodies. These models are less complex than the surface model, but they contain large errors near the surface of the body in study. These models require volume integrals, which become much more cumbersome than the surface integrals used in surface models.

Although the polyhedron model presents a strong use case for irregular bodies, it is not justified for use in modeling Earth's surface because Earth orbiting satellites rarely penetrate the

brillouin sphere. Further, the polyhedron model is slow as it requires a summation of the entire surface to obtain a force value. However, research suggests that there could be “utility in superimposing polyhedron and conventional spherical harmonic expressions in a planetary gravitation field, to include details such as ocean trenches, mountain ranges, or density variations” [8].

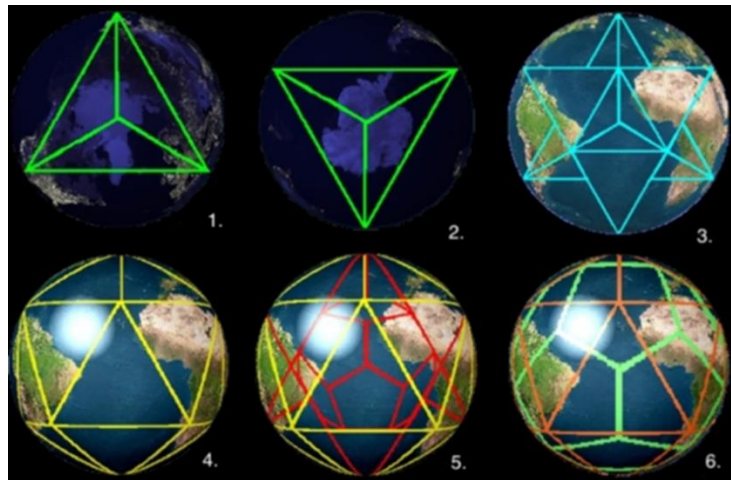


Figure: Icosahedron Earth grid subdivision [11]

4.2 Optical Modeling

Another useful feature of polyhedron modeling arises from its usefulness in the analysis of optical data. Imagery from spacecraft fly by or ground based doppler imaging telescopes can provide information regarding asteroid shapes. For example, JPL produced a paper using a shape model of asteroid 4769 Castalia that was derived from data collected by the Arecibo radio antenna [8]. In a comparison of the polyhedral model to the mascon and spherical harmonics models, the polyhedral model proved to be significantly more accurate close to the surface of the asteroid.

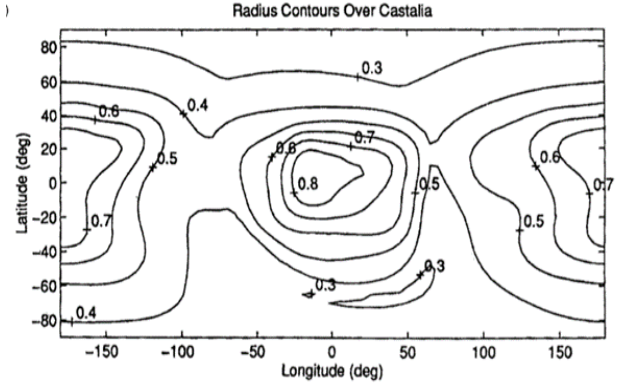
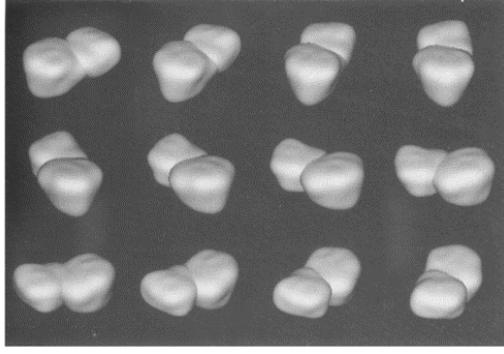


Figure: Digital contour mapping of 4769 Castalia Optical data [8]

4.3 Global PMC Model

Method

A global point mascon model (PMC) has been proposed as a solution to the cumbersome nature of the spherical harmonics [12]. The model places an arbitrary number of mass concentrations along Earth's surface to estimate the geopotential. By fixing the location of these mascons, the weighted linear least squares problem can be used to define a mass for each point.

$$J = \sum_{i=1}^m w_i \varepsilon_i^2 = \sum_{i=1}^m w_i [U_{PMC}(\mathbf{\eta}_i) - U_{SH}(\mathbf{\eta}_i)]^2 \quad (1)$$

where there are m measurements, ε_i and $\mathbf{\eta}_i$ are the residual and location of the i^{th} measurement respectively, U_{SH} is the measurement and is the potential evaluated according to SH, U_{PMC} is the measurement model and is the potential according to the point mascon model:

$$U_{PMC}(\mathbf{r}) = U_{2B+J_2} + \sum_{j=1}^n GM_j / |\mathbf{r} - \mathbf{p}_j| \quad (2)$$

where there are n mascons, GM_j is the gravitational parameter for the j^{th} mascon, \mathbf{p}_j is the location of the j^{th} mascon, and U_{2B+J_2} is the potential due to the two-body plus J_2 terms³⁷:

$$U_{2B+J_2} = \frac{GM_E}{r} \left[1 - J_2 \left(\frac{R_E}{r} \right)^2 \left(\frac{3z^2}{2r^2} - \frac{1}{2} \right) \right] \quad (3)$$

Figure: Global Point Mascon Equations and Least-Squares Problem [12]

Mascon Distribution

The lateral distribution of mascons is chosen using Thompson's problem which provides a solution that balances high resolutions without eliminating the useful gravity signature of each element. Further, the global radius of the mascon shell is chosen using an optimization loop that provides a bury distance that minimizes the performance index of the least squares problem.

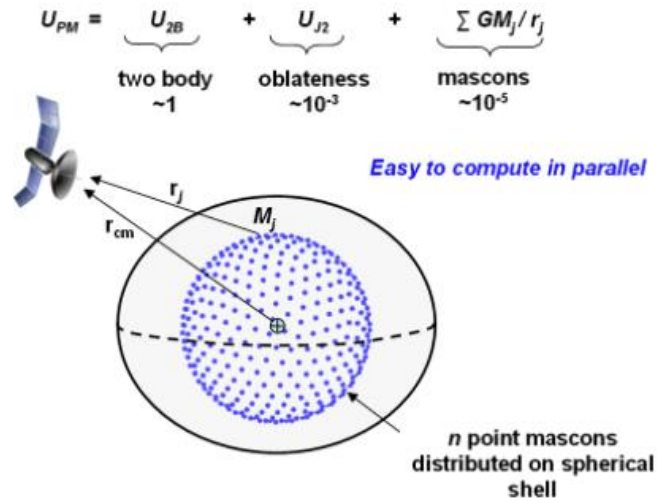


Figure: Sample mascon distribution schematic [13]

Results

Russel and Arora found that the choice of summation method is directly related to the precision of solutions. They suggest that an ill conditioned inversion results from mass terms that are both negative and positive and span throughout large orders of magnitude. To alleviate precision loss caused by the ill inversion problem, Russel and Arora suggest two summation methods: a Kahan approach and a divide and conquer method. They noted that a pairwise or cascade summation which incorporates recursive implementation to perform a naïve sum with a minor overhead could produce 1-2 digits of precession using ~ 6 recursive calls. This method

performed at a speed that was within a few percent of the naïve sum. In Russel and Arora’s tests, an ascending order sort of the mascon magnitudes was found to be preferred to other sort orders. However, they argue that further research could improve the summation by refining the sort order of the summation method. The “divide and conquer” method is deemed amenable to parallel computation and is expected to match CPU precision levels when implemented in GPU.

The required runtime for the mascon model is determined by the time required to solve the linear least squares problem. Russel and Arora argue that this runtime equivalent to the ratio of the number of mascons to the number of measurements provided (γ).

A second scaling parameter (α) is introduced to relate the number of mascons to the resolution of the SH fitting function. This parameter provides a relationship to determine runtime complexity given below.

$$d = \text{int}\left(\sqrt{n'/\alpha}\right)$$

The complexity of the computation grows with the fourth power of the parameter d which is equivalent to the size of the SH fitting function. The maximum value computed in Russel and Arora’s study was limited by computer memory at $d=156$, requiring 1 CPU day to complete. However, this complexity can be dramatically reduced as d is reduced.

Resolution of the mascon model is determined by how fine the mascon grid is. However, current mascon solutions are limited by the memory requirements for adding mascons. The following chart shows that the number of mascons (corresponding to memory use) grows exponentially with increased degree/order:

Model descriptor	Size of SH fitting field ($d \times d$)	No. of mascons (n)
PMC11n240	11×11	240
PMC33n1920	33×33	1920
PMC71n7680	71×71	7680
PMC156n32720	156×156	30,720

Figure: Point Mascon Models and their Corresponding Fields [13]

Russell and Arora suggest that further research ought to provide a method to fit mascons directly to geodetic measurements rather than fitting the SH function. This method will be employed in the RL05 model discussed below.

5. Linear Combinations of Functions Using Splines, Kernels, or Finite Elements

As discussed earlier, representations of the gravity field often break down using spherical harmonics. Although the solution of Poisson's equation can yield convergent solutions for the geopotential near the surface of nearly spherical bodies, these solutions are often inaccurate for irregular bodies. To alleviate these issues, Werner and Scheeres proposed the use of summations over the faces and edges of polyhedral models. However, these models are often computational intensive, making them unfit for Monte Carlo analysis and other trajectory planning procedures.

5.1 Finite Element techniques

5.1.1 Junkins

Method

The interpolation method for modeling the geopotential was proposed by Junkins in 1976 [14]. This method can be classified as a finite element approach that uses a weighting function to ensure a piecewise continuous approximation for the geopotential. Junkins provided

computational experiments proving that a weighted least squares approach was superior to Taylor series approximations in terms of RMS acceleration errors.

Junkins' model had the following criterion:

- 1) Finite model for shell within 1.2 Earth radii
- 2) Second order terms represented using Spherical Harmonics
- 3) Element sizing such that errors are in the seventh significant digit
- 4) Fixed order of local approximation of 3

Junkins replaced Taylor's series approximations with a judicious choice of Chebyshev approximations to replace high order polynomials with lower order representations. The following graph from his numerical experimentation shows that lower order Chebyshev functions can achieve accuracies comparable to Taylor's functions of higher order.

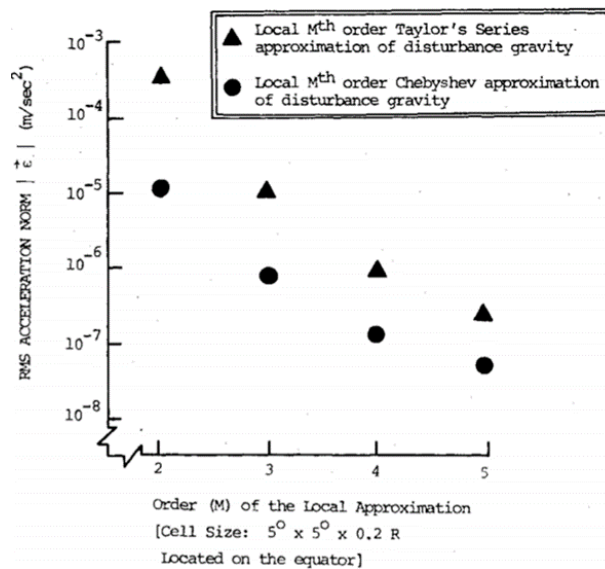


Figure: Order of local approximations vs RMS acceleration approximation error [14]

Results

Using these criterion, Junkins produced a 3rd order approximation of the geopotential containing 1500 finite elements. When compared to a 23 degree/order SH series, Junkins' model had a mean acceleration error of at least one order of magnitude lower than that of the SH model. His max acceleration error was also 4 times less than that of SH.

Junkins model exhibited a computational speedup of an order of magnitude by replacing a 23rd order SH expansion with a 3rd order weighted least squares problem. Junkins admitted that further research would be necessary to reduce the number of random accesses necessary to perform his calculations.

5.1.2 Colombi

Colombi's research proposes an interpolation scheme with adaptive local representation to compute gravity forces directly [15]. Colombi's interpolation scheme is closely linked to the interpolation methods proposed by Junkins, but Junkins model relies on a nearly spherical body of study. Colombi's method precomputes the domain of gravity forces, rather than potentials to and provides an adaptive spatial structure to yield a speed of 2 orders of magnitude relative to previous polyhedron interpolation methods.

Method

Colombi's model chooses to interpolate the force directly to improve the speed of the computation. His model uses cubic cells for the interpolation and uses Gauss-Lobatto-Legendre interpolations (GLL) with barycentric forms of the Lagrange polynomials. The low Lebesgue constant for this interpolation scheme ensures a relatively uniform approximation of the interpolation function. Furthermore, the barycentric form of the Lagrange polynomials is chosen for computational efficiency. The polynomial approximation appears as a linear combination of

function values at the interpolation nodes, which corresponds to a component of the gravitational force.

The adaptive local representation is implemented using an octree data tree structure. This structure subdivides each elemental cube in the model until the element meets a required error tolerance. The benefit of the octree structure is that provides higher computational speed than traditional polyhedral subdivisions. In fact, the octree structure provides a sub-linear relationship between run-time complexity and the number of subdivisions required (corresponding to the resolution or minimum size of each element), whereas polynomial divisions are linear. The result of this relationship is that the octree model will provide more speed than the polyhedral model, and this performance enhancement will scale with increasing numbers of subdivisions. Since the octree structure is inherently bounded, spherical harmonics are used to calculate the gravity field outside of the octree structure.

The following schematic of a quadtree portrays the 2D representation of the octree. It is notable that this structure allows for adaptive resolution by allowing the algorithm to make subdivisions only where necessary.

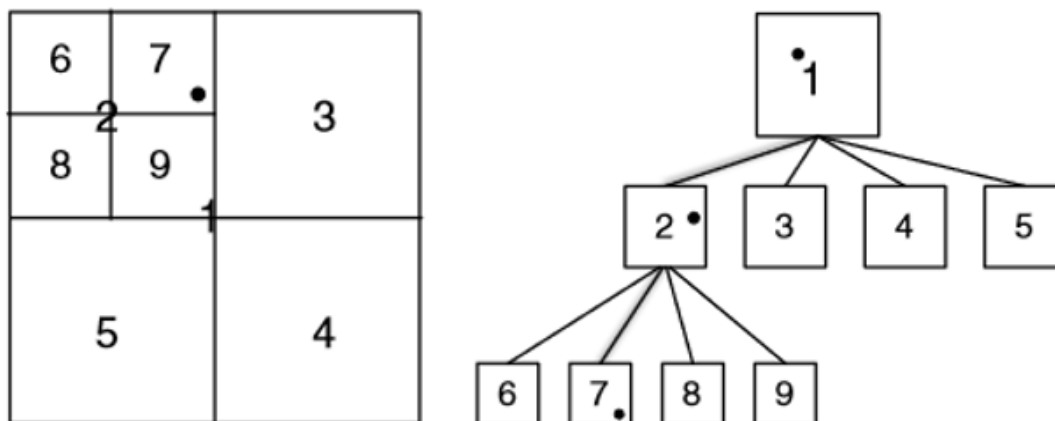


Figure: Quadtree schematic representing adaptive query resolution [15]

Example Octree Construction and Performance

The 1998 ML14 asteroid was studied using an octree with 10 layers, corresponding to a smallest cell size of 4.8 meters. This cell size was chosen from experiments that determined a required resolution to minimize error in modeling the asteroids curved surfaces. The highest order polynomials required for the octree were of order 6.

To refine the structure, each cell was tested at 10,000 sample points so that the subdivision would produce an error below $5e-7$. The octree construction required 1150 CPU hours on a message passing interface (MPI). The process required 64 processors, 18 hours of clock run-time, and 635 MB of memory. Outside the octree bound, spherical harmonics of degree/order 12 were employed.

Results

The performance of the various force models in generating several trajectories was analyzed against a reference trajectory that used Werner and Scheeres' polyhedron model with a required error below $10e-10$.

A. Close Retrograde Orbits

111 trajectories were analyzed, and most of these propagations fell within 2 meters of the reference trajectory. The max position error was 2.56 meters. The cubetree was 112 times faster than the augmented polyhedral method.

B. Midrange Orbits

The initial conditions for the midrange orbits were placed at 1250 meters and 1500 meters from the center of the transition between octree and spherical harmonics representations. 1487 trajectories were analyzed, where most differed by less than 2 meters and the max error exhibited was 10.24 meters. The cubetree method was 90 times faster than the augmented polyhedral method.

C. Random Trajectories

The initial conditions for the random trajectories ranged between 600 and 1500 meters. In 911 simulations, only 38 trajectories had an error greater than 2 meters, and 8 trajectories had errors larger than 100 meters. The cubetree method was 111 times faster than augmented polyhedral.

D. Ejecta Trajectory

Of the ejecta trajectory simulations, only one test showed errors larger than .5 meters. These trajectories were calculated 169 times faster than the polyhedral method.

The 15 trajectories with the worst-case errors were analyzed in the frequency domain using FFT transforms. A sensitivity analysis using a Monte Carlo simulation of perturbations around the initial conditions for these orbits showed large variations in the resultant trajectories under small perturbations. This analysis showed that the dynamics of these trajectories were highly sensitive.

The adaptive spatial structure with polynomial interpolation proves to be an accurate tool for performing trajectory analysis and mission planning near irregular bodies.

Future Work

Colombi suggests that further work could improve the adaptive local representation technique in his model. For example, he notes that the construction of interpolation points using cartesian products uses more points than required for the given accuracy. Further, he notes that this representation is not globally exact, as it interpolates forces rather than gradients of potential. He suggests that this could be improved using a Hermite interpolation or other techniques, chosen after rigorous study of the impacts of the change.

Colombi also mentions that his model creates discontinuities across cell boundaries. Although his experimentation did not find these discontinuities to impact his trajectory propagation, he notes that they could impact other orbits like those of low thrust trajectories. He suggests that regularly subdivided tetrahedra or other octree variants could provide for continuity.

Further research should also explore different techniques for capturing interpolated values. Poisson's equation or physical experiments could provide starting points for these endeavors.

5.2 Cubed Sphere Model

Beylkin and Cramer suggested mapping the sphere to a cube to compute the geopotential and acceleration. Each face is segmented with uniform grids and multiple concentric shells are mapped to these cubes to perform function interpolations. Beylkin and Cramer suggested three interpolation methods [16]:

1. B-splines defined on the surface of the sphere
2. Polynomials on subdivisions of the surface of the cube
3. B-splines on the surface of the cube

The first method suffers from stretching that is caused by oversampling near the poles. The second method does not experience stretching, but it requires a much higher sampling rate than B-splines. Choosing the third method, which employs B-splines on the surface of the cube does not suffer from stretching or require oversampling. This method, known as the cubed-sphere model was determined superior as a technique that trades speed for memory.

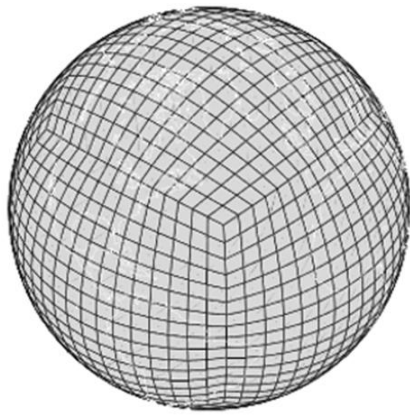


Figure: Illustration of the mapping from a sphere to a cube [17]

Jones' refined the Junkins' theory in a model that uses the GGM02C model as a "truth model" for comparison. Using multiple concentric shells and Chebyshev basis functions, allowed for speedups on the order of 30 for the 150x150 model. The breakeven resolution for speed occurred for the 20x20 model. Jones' model had only required 856 MB of memory. Agreement between SH and cubed-sphere model broke down under 200 km, where the cubed-model outperformed SH in some cases due to the fluctuations in the integration constant at the limits of machine precision. Furthermore, the cubed sphere model reduced gravitational anomalies.

Jones suggests that the cubed sphere model should be further refined for moon-based models to account for large mass concentrations created by asteroid impacts. Furthermore, the model requires additional tuning below 50 km of altitude. Finally, Jones argues that further research

could provide a useful implementation for the cubed sphere model in orbital determination procedures with the addition of integration with non-linear filters, like those of the unscented Kalman filter.

Cubed-sphere model	Spherical harmonic model	Average speedup factor
CS-30	20×20	0.73
CS-76	70×70	5.97
CS-162	150×150	30.82

Figure: Speedup factors for varying degree Cubed-Sphere Models [17]

Model	Position, mm				Velocity, mm/s			
	Min	Max	Mean	Median	Min	Max	Mean	Median
CS-30	4.70×10^{-5}	0.0200	0.0042	0.0036	2.99×10^{-8}	2.27×10^{-5}	4.84×10^{-6}	4.11×10^{-6}
CS-76	3.25×10^{-5}	0.0177	0.0041	0.0033	3.22×10^{-8}	2.04×10^{-5}	4.73×10^{-6}	3.79×10^{-6}
CS-162	5.71×10^{-5}	0.0176	0.0040	0.0033	2.50×10^{-8}	2.04×10^{-5}	4.64×10^{-6}	3.80×10^{-6}

Figure: Cubed-Sphere state 3-D RMS performance at 300 km [17]

5.3 AstroHD

AstroHD provides “a revolutionary digital approach...to three-dimensional dataset storage and modeling, particularly for gravitational, magnetic field, and atmospheric data.” The method aims to leverage the benefits of cheaper and highly capable random-access memory (RAM). The digital approach increases computational speed and precision while producing complete, high order fields that do not suffer from truncation error like spherical harmonics.

Method

AstroHD seeks to minimize the effort needed to perform interpolations. In doing so, the software minimizes the number of grid points required for a desired angular resolution. Additionally, these grid points are constrained such that all data is weighted equally. The employment of 3D dataset storage for icosahedron points provides an efficient method for mapping grid points. Furthermore, a unique mapping technique is employed to reduce the number of grid points.

AstroHD replaces the inefficient right ascension/declination spherical coordinate system with a digital icosahedron representation. The subdivided icosahedron yields a perfectly symmetrical collection of non-planar points. The subdivision used a linear interpolation of primary faces and great circle arcs to provide a uniform set of cells that maintain the same shapes. The digital dataset led to 40 percent less function evaluations and memory usage than spherical harmonics representations. Furthermore, a variable step size yielded computational accuracy gains by de-emphasizing the importance of gravity perturbations at high altitudes.

Results

Numerical experimentation compared the AstroHD representation to a 145x145 EGM96 spherical harmonics base-model. The numerical experiments were run using 7 icosahedron shells separated by 5 km. Three tests were run, with vertex to vertex angular separations of 1, .25, and .1 degrees. These grids required 1 minute, 50 minutes, and 4 hours of computational time, respectively.

The finest grid, (.1-degree angular separation), required ~4 million functions and evaluations or 46 MB of storage. Whereas this memory footprint is relatively small, a field with

more higher resolution data at varying altitudes would require much more memory. For example, a field with 100 icosahedron shells at the same angular resolution would require 4.6 GB of memory.

The numerical experiment also performed sample orbital integrations to compare the AstroHD method to spherical harmonics representations. The AstroHD performance yielded high speed and accurate solutions for 10-day propagations. Propagation throughout the coarsest grid (1-degree angular separation) yielded a max error of 3 km, which is comparable to a truncated 40x40 spherical harmonics representation. The grid with angular resolution of .25 degrees exhibited an error of 1 km over 10 days, which is comparable to the 90x90 SH solution. Finally, the finest (.1 degree) grid yielded a max error of 35 meters. The following graph depicts the relative accuracy of each AstroHD field in relationship to varying degree spherical harmonics.

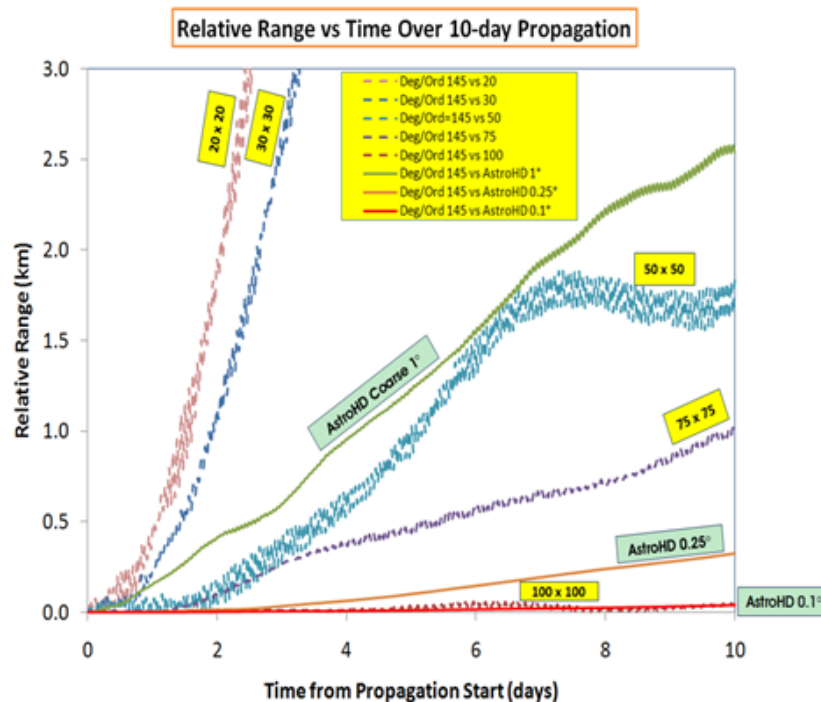


Figure: Comparison or residual errors between SH and AstroHD orbit propagations [2]

The following graph relates the runtime of the AstroHD propagation to that of the spherical harmonics solution for varying degree. The data shows that AstroHD provides a speedup on the order of 20 compared to the 125x125 SH field. Furthermore, the data indicates an accuracy improvement of 620 times for a spherical harmonics field of the same runtime.

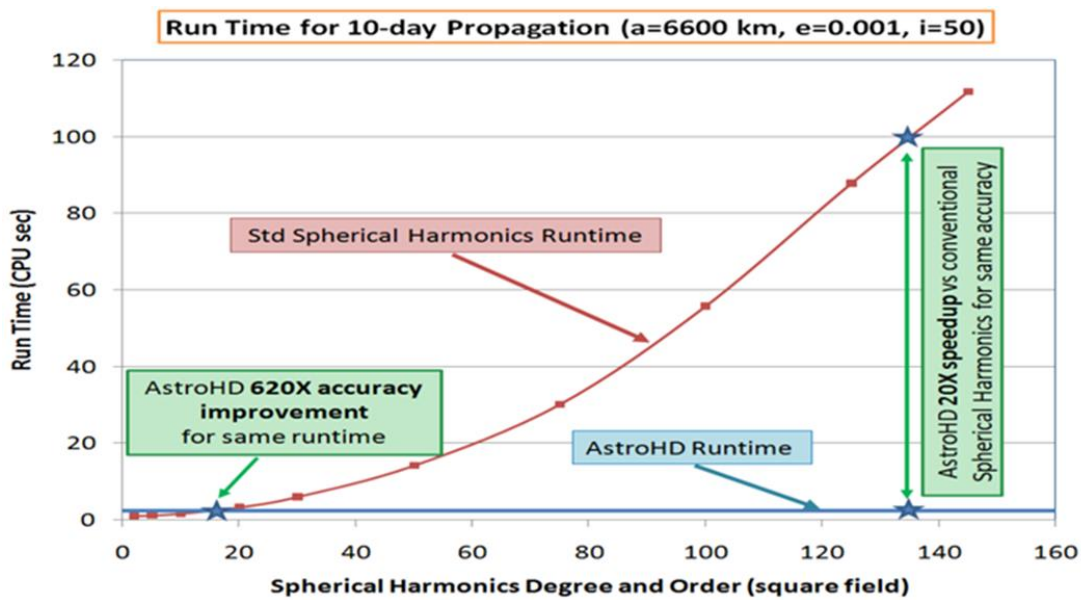


Figure: Run time Comparison for SH and AstroHD propagations [2]

5.4 FETCH

The fetch interpolation model [7] uses the typical Junkins weighting function to force continuity and smoothness for its solutions. Fetch is unique because it uses an overlapping grid strategy and an adaptive order-based method for selecting polynomials that minimize memory requirements. Furthermore, the method allows for analytic inversion of the normal equations that eliminate the need for linear systems solvers.

The benefits of fetch lie in its ability to quickly produce continuous and smooth models that are singularity free. The unique grid strategy of the method minimizes memory costs and adaptive polynomial selection minimizes the memory required to store coefficients. Finally, the analytic inversion method allows for very rapid solutions to the least-squares problem and orders of speedup compared to prior interpolation methods. Finally, Arora and Russell's analysis of fetch prove that it is highly accurate, producing residual errors that are within the noise of the spherical harmonics underlying model.

Fetch Results

Four Fetch models were produced from the GGM03C field. The highest resolution field was computed using a 360x350 GGM03C SH field, which took 12,000 CPU hours and 2.36 GB of memory to complete. Fetch offered speedups on the order of 3 orders of magnitude, with runtime matching SH for an 8x8 field. The major advantage of the Fetch model allowed it to compute gradients and higher order partial derivatives directly with no additional memory costs. The minimum breakeven resolution, compared to SH, was determined to be degree/order of 360. The models are continuous up to order 3 and 60 Earth radii. The memory requirements and speedups are listed below:

Table 2 Fetch models										
Model name	SH field	η_1	η_2	η_3	η_4	Γ	Memory, MB	No. cells	Average no. coefficients/cell	Expected speedup
F399m33v1	33 × 33	1.00	10.00	11.00	10.00	2.00	121	175,536	90.2	3
F399m70v1	70 × 70	1.00	10.00	11.00	11.00	1.53	360	564,075	83.1	15
F399m156v1	156 × 156	1.00	12.00	11.00	11.00	1.21	898	2,015,076	57.9	91
F399m360v1	360 × 360	1.00	20.00	11.00	8.00	0.92	2360	7,615,254	41.0	784

Figure: Performance characteristics of varying degree Fetch models [7]

6 Parallel Computation

6.1 CUDA

Method

The implementation of the Global Point Mascon Model in compute unified device architecture (CUDA) can achieve one order of magnitude speedups compared to traditional computational methods. Previously, heterogenous computing architecture implemented PMC using multi-core processors and accelerated coprocessors. However, CUDA enables the use of CPU as the host device in tandem with GPUs from NVIDIA that support parallel implementations due to their ability to carry double precession operations.

The basic implementation of PMC in CUDA uses the CPU for logical tasks while the GPU provides parallelism. The fundamental unit of the GPU is a thread, a collection of threads form blocks, and the blocks create the grids necessary for tasks. All the threads which belong to one grid then execute the kernel function. The CUDA provides a layered memory space comprising of global, local, shared, texture, and registers.

The implementation of the PMC with parallel computation based in GPU is relatively straightforward. Since the most time intensive portion of PMC modeling occurs during the computation of the gravity forces at the target point, implementing this computation in parallelism with the GPU provides the most opportunity for speedups. The following list provides a simplified pseudocode for the operation:

1. Each block loads information about the mass points from global memory and saves it in shared memory

2. Each thread in the block uses the point info to compute gravitational acceleration at the target point, generated by the single point, and saves this information to shared memory
3. When all threads complete the computation, the reduction method sums all results in the block to a single value and save it to shared memory
4. The new kernel function is launched to sum the results of all blocks producing the resultant gravity field

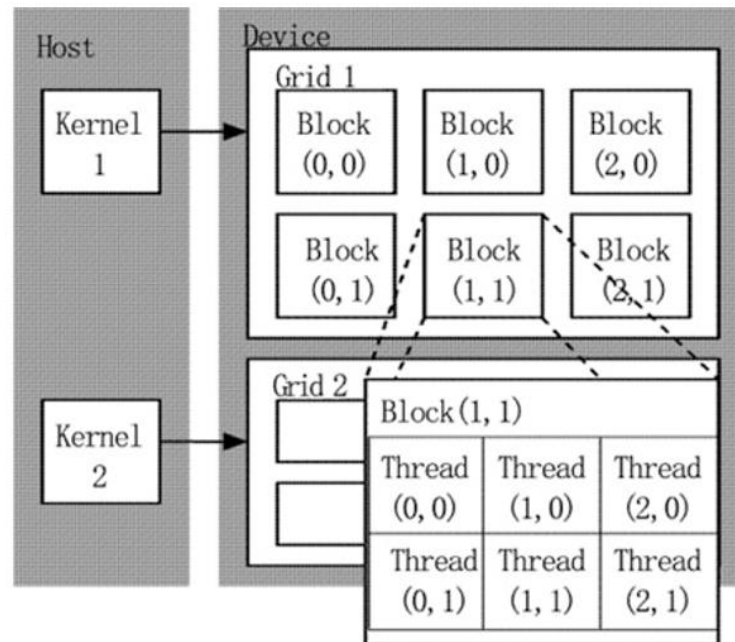


Figure: CUDA's heterogeneous programming [13]

Results

The CUDA implementation was tested in a simulated scenario. The scenario used a spacecraft trajectory model using a Runge-Kutta, Dormand-Prince numerical integration of Earth's gravity. The results of this integration were compared to the SH model. Speedups for low order fields (~30) were on the order of 5, and speed ups for the order of 70 were on the order of 8. The global PMC model was 10 times faster than spherical harmonics for high order

gravitational fields. Furthermore, CUDA met the precision requirements (error of about .01 m).

In fact, the accuracy of the CUDA method were on the same order as the SH model. CUDA required 3 days of integration time to achieve max residuals of 2 cm for LEO and 2 mm for HEO. The following table summarizes these results:

Orbit type	Max error (m)	Speed-up ratio
LEO	0.0167	15
MEO	0.0020	11
GEO	0.00065	14
Molniya	0.0020	15

Figure: Error and speed-up ratio for CUDA propagation of different orbit types [13]

7. Conclusion

The traditional method for computing the gravity field, spherical harmonics no longer fulfils the evolving requirements of gravity estimation and determination applications. Although the method provides a very high-fidelity representation of the field with low memory requirements, its low speeds are not suitable for many applications. Existing spherical harmonics models remain relevant as base models for the “fitting” of new models. Numerical experimentation allows for the reduction of residuals in new models which aim to match the accuracy of spherical harmonics within given tolerances. In fact, spherical harmonics remains as the only exact representation of the gravity field in its infinite form.

The major conclusion of this academic survey is that gravity fields can be quickly computed through efficient discretization of the field (in contrast to the continuous nature of the

SH equation). The strategies listed in this paper achieve this discretization through a variety of forms:

1. Polyhedral grids with a series of grid points or planar surfaces
2. Grids of point mass concentrations
3. Finite element techniques which generate piecewise continuous, localized approximations
4. Basis splines

Discretized models allow for computational advantages by allowing for innovations in:

1. Adaptive local representations
2. Adaptive polynomial interpolation
3. Parallel computation
4. Creative storage datasets

Although some of these innovations have been tested for particular discretization methods, it is important to note that these improvements can apply elsewhere. For example, adaptive representations could benefit mascon solutions or creative datasets could be employed for mascons rather than icosahedron subdivisions.

8. Applications

It would be naïve to conclude this analysis by proclaiming a superior method for representing the gravity field. Instead, each strategy exhibits different relative strengths and weaknesses depending on the requirements of its applications. The following list will attempt to define the requirements of some of the current applications of gravity field estimation and suggest an ideal gravity representation for that application.

A. Irregular Bodies

Irregular bodies require a model that can maintain its integrity for objects that are far from spherical. Additionally, spacecraft missions to asteroids require on-board computation of the gravity field for objects of which little a-priori knowledge exists. Several techniques for doing so are discussed in this paper. In general, it is important to make use of optical data to provide a baseline for asteroid surface information. Data from the optical surface model can then be used to subdivide the irregular body into a set of planar surfaces or grid points (like those of the icosahedron). These surface representations can then be used to compute combinations of potential functions or finite element techniques.

B. Earth Sciences

Earth sciences require high fidelity fields that can represent very small gravitational perturbations related to density variations and the non-spherical nature of Earth's surface. Additionally, Earth science require capabilities to study the temporal variations associated with changes in hydrology. The spherical harmonics solution is best fit for Earth science's studies which require high accuracy gravity data.

C. Trajectory Planning/Monte Carlo

Trajectory planning and Monte Carlo simulation require high speed estimation for propagation of up to thousands of orbits. Additionally, relatively high levels of accuracy are required within the requirements of the mission. The implementation of polyhedral or mascon modeling with creative datasets and adaptive resolution or adaptive interpolation have been shown to be well suited for these applications.

D. On-Board Computation

On-board computation can refer to gravity estimation or propagation integrations that occur on a satellite during flight. This application is unique in its restrictive hardware requirements. On-board computation requires relatively low memory requirements, high speed, and varying degrees of accuracy. On-board applications require portable, yet highly efficient, processors. The parallel computation techniques discussed in this paper require further research for on-board computation but are well suited for future endeavors.

9. Future Research

This paper discusses a variety of techniques that seek to benefit the computation of the gravity field. It is notable that these techniques are not mutually exclusive, and some strategies should be combined to create a comprehensive gravity model. For example, it is suggested that the polyhedron and spherical harmonics solutions be superimposed to provide better representations surface features like mountains and valleys on bodies.

Most of the research presented in this paper describes the numerical experiments which aim to characterize models based off their relationship to spherical harmonics base models. Additionally, many of the models are configured by minimizing the residual errors between the new model and SH. Future research should seek to derive the developing modeling techniques directly from measurements, rather than in relationship to SH. Additionally, this research should seek to move past numerical experimentation by employing the models for their intended use.

Although it is beyond the scope of this research, the refinement of integration techniques is required for high speed orbit propagation in trajectory planning or other orbital propagation uses.

The refinement of gridding techniques like those of the polyhedral models, icosahedron subdivisions, or mascon spacing provides an opportunity for speed improvements and memory reduction. For example, a systematic method for including adaptive resolution within mascon models could prove useful for ground track analysis. Further, the polynomial representations in this paper assume that each discretized point or surface has a constant density. Although this assumption has not been shown to cause large errors, a variable density technique could prove useful.

The final argument of this research is that the continuous refinement of parallel computation techniques is the most valuable opportunity for the future of gravity estimation and determination. Parallel computation offers an opportunity for “divide and conquer” approaches while maintaining portability in the form of common GPU hardware. Cards like those used for the CUDA experimentation could benefit stationary and on-board computations alike.

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